ADOLPH BASSER
COMPUTING LABORATORY

SILLIAC CODE R6

TITLE Fast Square Root.

TYPE Closed. All modified orders re-set before exit.

NUMBER OF WORDS 19.

TEMPORARY STORAGE 0, 1, 2.

ACCURACY See below.

DURATION For numbers uniformly distributed in the range 0 to 1, the average duration is 5.6 milliseconds (R1 averages 6.0 milliseconds). For very small numbers R6 is much faster than R1, the maximum duration being 9.0 milliseconds.

PURPOSE This routine is intended for use in place of the standard square root routine R1 in those programs where time taken by the square root routine is of more importance than the store space occupied by it. The time saved by using R6 is appreciable only for numbers less than 1/10. The graph on page 3 provides a comparison of the times taken by R1 and R6 to find the square root of numbers of various magnitudes.

DESCRIPTION R6 is entered with the normal subroutine entry with a positive number 'a' in the accumulator. It computes 'x', the square root of a, and links back with x in the accumulator. If entered at the right hand order in the first word it computes the square root of the number in store location 1.

R6 is not designed to handle negative numbers. If 'a' is negative, R6 may loop or may produce an answer x which may or may not be the square root of -a. What happens depends upon the magnitude of a. If a negative value of a is liable to occur, a suitable test should be included in the main program. Alternatively, R6 may be entered with a 36 order in place of the usual 26 order, in which case a test for negative x should be made.

MATHEMATICAL METHOD

R6 first computes

$$a' = a2^{2r}$$

where r is such that

$$1/4 \leq a' \leq 1,$$

and then computes successive approximations x_1 to x_n by Newton's method. If we start with

$$x_0 = 2^{-r}$$

the next iterate is

$$x_1 = 1/2(a/x_0+x_0)$$
$$= (a' + 1).2^{-r-1},$$

which is computed by R6 directly from a' and r. The successive iterates are calculated from the formula

$$x_{n-1} - x_n = 1/2(x_{n-1} - a/x_{n-1})$$
$$= e_{n-1}.$$

The x_i forma a monotonically decreasing series which converges to the true root. The Convergence is of second order, hence, if e_n is small it provides a good estimate, albeit an underestimate, of the error in x_n. The iteration ceases at x_n when

$$e_{n-1} \leq 2^{-20},$$

for then we know that

$$e_n \leq 2^{-41}/x,$$

and this is the error which is inherent in x, the "exact" root of 'a', owing to the fact that 'a' is known only to a total of 39 binary digits.

However, there is an additional error in x_n due to the division round-off which, for sufficiently large values of x, predominates over the error calculated above. Thus we have:

If $a > 1/16$

$$-2^{-39} \leq x_n - x \leq 2^{-40}$$

whence

$$-2^{-39}x \leq a - x_n^2 \leq 2^{-38}x.$$

If $a<1/16$ fewer than 39 digits are needed to express x accurately and hence the division round-off is insignificant and

$$-2^{-40} \leq a - x_n^2 \leq 2^{-40}.$$

CODED BY: B.A. Chartres.

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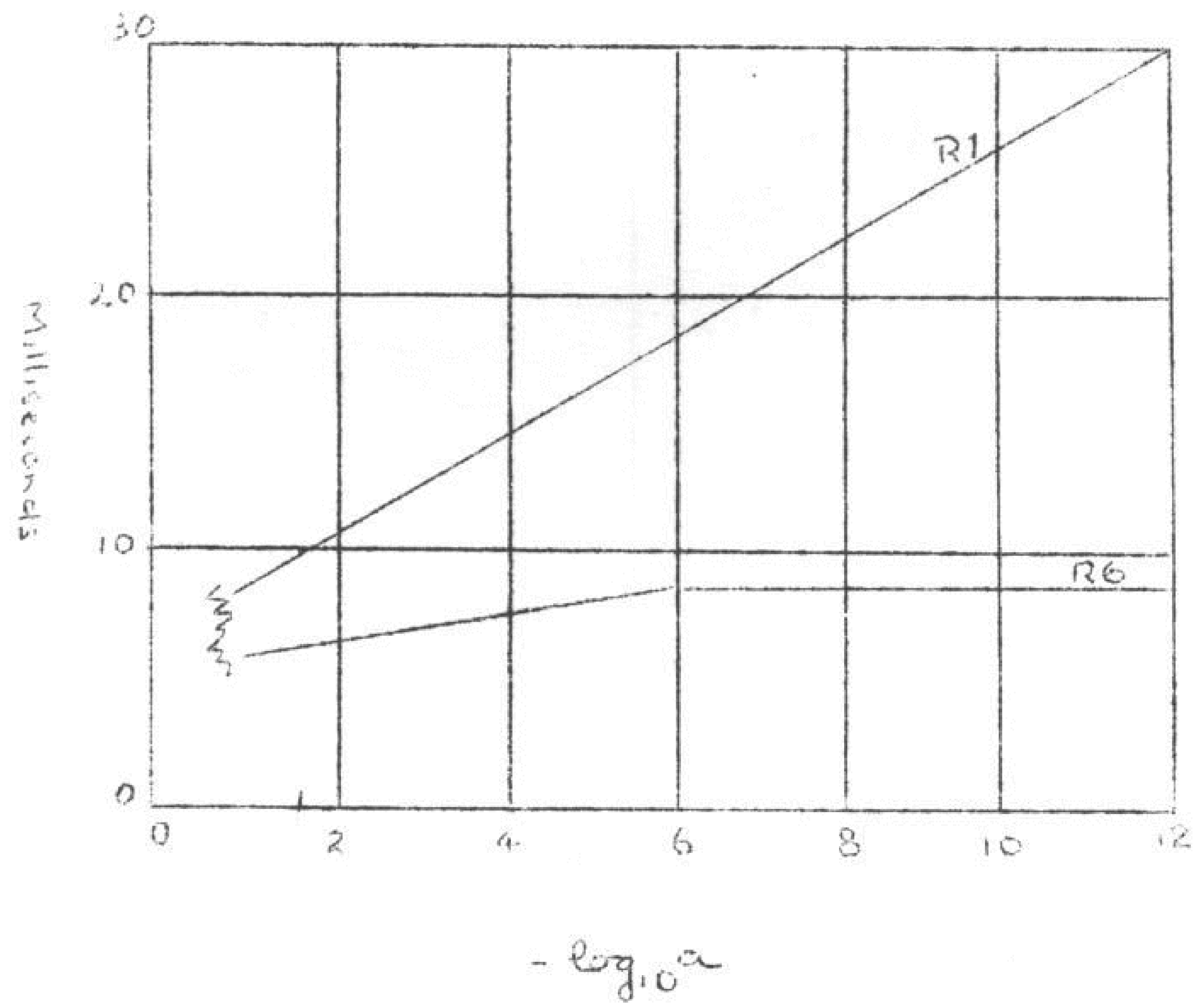
APPROVED BY: J.M. Bennett.



LOCATION	ORDER		NOTES
0	40 1F		Store a in 1.
	L5 16L		
1	40 F		Plant link in 0.
	K5 F		
2	42 F		
	50 1F		a $\longrightarrow$ (Q).
3	S3 F		Link out with (A)=0 if a=0.
	32 F		
4	S5 F	From 7	
	L4 17L		
5	32 7L		Test if $(Q) \geq 1/4 - 2^{-39}$
	00 2F		Scale up a by factor 4.
6	F5 8L		Prepare shift order.
	42 8L		
7	26 4L		
	L5 17L	From 5	$(A)' = 2^{-39} - 1/4, (Q) = a'$.
8	42 8L		Re-set shift order.
	10 (1)F	By 6',8.	$(Q)' = x_1 = (a'+1).2^{-r-1}$.
9	S5 F		
	40 2F	From 15.	$x_i \longrightarrow (2)$
10	50 2F		
	L1 1F		
11	66 2F		$(Q)' = a/x_i$.
	S7 F		
12	L0 2F		$(A)' = \lvert a/x_i \rvert - x_i$.
	10 1F		$(A)' = x_{i+1} - x_i$.
13	L4 18L		
	32 15L		Test if $\lvert x_{i+1} - x_i \rvert \leq 2^{-20}$
14	L0 18L		
	L4 2F		$(A)' = x_{i+1}$

LOCATION	ORDER		NOTES
15	22 9L		
	L0 18L	From 13'	
16	L4 2F	By 0'	Link out.
	22 F		
17	F0 F	By 4',7'	$-1/4 + 2^{-39}$
	00 1F		
18	00 F	By 13,14, 15'	2^{-20}.
	80 F		